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B.A./B.Sc. THIRD SEMESTER EXAMINATION, DECEMBER 2024
SECOND YEAR (BATCH 2023-27)
MATHEMATICS (HONOURS)
Paper : 3MTMMJC1

Date : 02/12/2024
Time : 11.00 am – 1.00 pm
Full Marks : 50

Group - A

Answer all the questions. Maximum one can score is 25.

1. a) For a group G, show that ${}^{G}\!/\!_{\mathbb{Z}(G)}$ is never a nontrivial cyclic group.

   b) Does there exist a surjective homomorphism from the group $\mathbb{Z}_6$ onto $\mathbb{Z}_4$? Justify. [4+3]

2. Justify whether the group G is an internal direct product of two proper subgroups.

   a) $G=\mathbb{Z}_8$ b) $G=\mathbb{Z}_{12}$. [2+2]

3. a) Show that upto isomorphism, there are only two groups of order 4.

   b) Suppose G is a noncyclic group of order 39. Find the number of elements of order 3 in G. [6+4]

4. a) Prove that if p is prime then a group of order p^2 is abelian.

   b) Let G be a group of order p^n where p is prime and $n>0$. Prove that any subgroup of G of order p^{n-1} is a normal subgroup of G. [4+5]

Group - B

Answer all the questions. Maximum one can score is 25.

5. Prove that the following two matrices are not row equivalent:

$$\begin{bmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix}.$$ [4]

6. Describe explicitly all 2×2 row-reduced echelon matrices. [2]

7. Find bases for the following subspaces of F^5(where F is any field) :

   $W_1=\{(a_1,a_2,.......,a_5)\in F^5; a_1-a_3-a_4=0\}$ and

   $W_2=\{(a_1,a_2,.........,a_5)\in F^5; a_2=a_3=a_4 \text{ and } a_1+a_5=0\}$.

   What are the dimensions of W_1 and W_2? [4]

8. Let V be the set of pairs (x,y) of real numbers and let F be the field of real numbers. Define

   $(x,y)+(x_1,y_1)=(x+x_1,0)$, and $c(x,y)=(cx,0)$; where $c\in F$.

   Is V, with these operations, a vector space? Justify your answer. [3]

9. Let W_1 and W_2 be subspaces of a vector space V such that the set theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces $W_i\,(i=1,2)$ is contained in the other. [4]

10. Find three vectors in $\mathbb{R}^3$ which are linearly dependent and are such that any two of them are linearly independent. [2]

11. Prove that elementary row and column operations on a matrix are rank preserving. [5]

12. Let $B=\{\alpha_1,\alpha_2,\alpha_3\}$ be the ordered basis for $\mathbb{R}^3$ consisting of $\alpha_1=(1,0,-1)$, $\alpha_2=(1,1,1)$, $\alpha_3=(1,0,0)$. What are the coordinates of the vector (a,b,c) in the ordered basis B ? [3]

14. Prove that the vector space $\mathbb{R}$ over $\mathbb{Q}$ is not finite dimensional. [3]

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